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Applications of the Generalized Knaster–Kuratowski–Mazurkiewicz Theorem to Variational Inequalities

CARLO BARDARO AND RITA CEPPITELLI

Dipartimento di Matematica, Università degli Studi di Perugia, Via Pascoli, 06100 Perugia, Italy

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INTRODUCTION

The aim of this paper is to present some generalized versions of the M. Lassonde results on variational type inequalities (see Proposition 1.4 and Theorems 2.2 and 2.4 in [22]) by using some extensions (Theorems 1 and 4) of the Knaster–Kuratowski–Mazurkiewicz Theorem.

Such a KKM approach was developed by K. Fan [8] who pointed out its numerous applications to several nonlinear problems [9, 10, 11, 12, 13]. And also, this method notably lent itself to solving variational inequalities (see [6, 4, 24, 22]). In particular, Lassonde proved general theorems on inequalities involving semimonotone functions which improve the results of H. Debrunner and P. Flor [5], P. Hartman and G. Stampacchia [16], and J. L. Joly and U. Mosco [20].

We extend the Lassonde results in various directions. The essential improvements consist both in removing the linear structure on the domain of the involved functions and in assuming an order complete Riesz space as the range of functions.

In this abstract setting, already considered in [2, 3], we introduce a general definition of concavity (H-concave and H^*-concave function), as well as a new concept of semimonotone and hemicontinuous functions. These definitions, which in our structure have been adapted, are natural extensions of the classical ones.

1. NOTATIONS AND DEFINITIONS

In this paper we discuss the general setting which has been introduced in [2, 3] and is specified by the following definitions.

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DEFINITIONS. An *H-space* is a pair $(X, \{\Gamma_A\})$ where X is a topological space and $\{\Gamma_A\}$ is a given family of nonempty contractible subsets of X, indexed by the finite subsets of X, such that $A \subset B$ implies $\Gamma_A \subset \Gamma_B$. Let $(X, \{\Gamma_A\})$ be an H-space. A subset $D \subset X$ is called *H-convex* if, for every finite subset $A \subset D$, it follows $\Gamma_A \subset D$. A subset $K \subset X$ is said to be *H-compact* if for every finite subset $A \subset X$ there is a compact, H-convex set $D \subset X$ such that $K \cup A \subset D$.

If $y \in X$ we set $K_y = \bigcap \{D \subset X\colon D \text{ is } H\text{-convex and } K \cup \{y\} \subset D\}$.

A subset $X_1 \subset X$ is called *compactly open* [*compactly closed*] if X_1 is open [closed] relative to every compact subset of X.

A multifunction $F\colon X \to X$ is called H-KKM if $\Gamma_A \subset \bigcup_{x \in A} F(x)$, for every finite subset $A \subset X$.

Given a multifunction $F\colon X \to X$ we put $F^{-1}(y) = \{x \in X\colon y \in F(x)\}$ and $F^*(y) = X \setminus F^{-1}(y)$.

Finally, let (E, C) be a Riesz space, where C is the positive cone, provided with a linear order-compatible topology; i.e., C is closed (see [14]). The interior of the cone C, denoted by $\mathring{C}$, will be assumed to be nonempty.

2. SOME GENERALIZATIONS OF CONCAVITY

Let $(X, \{\Gamma_A\})$ be an H-space.

Let $\Sigma^{(n-1)} = \{(t_1, ..., t_n) \in \mathbb{R}^n\colon \sum_{i=1}^n t_i = 1;\ t_i \geqslant 0,\ i = 1, ..., n\}$, be the $(n-1)$-dimensional simplex and let $e_1, ..., e_n$ be its vertices. If A is a finite subset of X with $\operatorname{card} A = n$, let $\beta_A\colon \Sigma^{(n-1)} \to \Gamma_A \cup A$ be a surjective function which maps each n-tuple of $\Sigma^{(n-1)}$ into an element of Γ_A such that $\beta_A(e_i) = x_i \in A$, $i = 1, ..., n$.

Given a function $f\colon X \to E$, we put

$$\mathscr{I}_f = \{(x, \lambda) \in X \times E\colon f(x) \in \lambda + \mathring{C}\}$$

and

$$\mathscr{I}_f^* = \{(x, \lambda) \in X \times E\colon f(x) \nleq \lambda\}.$$

DEFINITION 1. A function $f\colon X \to E$ is called *H-concave* [*H^*-concave*, respectively] if for every finite $A = \{x_1, ..., x_n\} \subset X$ and for every $z \in \Gamma_A$ there is an n-tuple $(t_1(z), ..., t_n(z)) \in \beta_A^{-1}(z)$ such that

$$(\Delta) \qquad \left(z, \sum_{i=1}^n t_i(z)\, \lambda_i\right) \in \mathscr{I}_f$$

$$(\Delta^*) \qquad \left[\left(z, \sum_{i=1}^n t_i(z)\, \lambda_i\right) \in \mathscr{I}_f^*, \text{ respectively}\right]$$

for every $(\lambda_1, ..., \lambda_n) \in E^n$ with $\{(x_i, \lambda_i),\ i = 1, ..., n\} \subset \mathscr{I}_f$ $[\{(x_i, \lambda_i),\ i = 1, ..., n\} \subset \mathscr{I}_f^*$, respectively]. Every n-tuple $(t_1(z), ..., t_n(z)) \in \beta_A^{-1}(z)$ satisfying (Δ) [(Δ^*) respectively] is said to be *compatible with the H-concavity of f [with the H*-concavity of f]*.

Remark 1. If $E = \mathbb{R}$, we have

$$\mathscr{I}_f = \mathscr{I}_f^* = \text{ipo } f = \{(x, \lambda) \in X \times \mathbb{R} : f(x) > \lambda\}.$$

So, for a topological vector space X, every concave function $f: X \to \mathbb{R}$ is both H-concave and H^*-concave. Indeed, let $A = \{x_1, ..., x_n\} \subset X$ and $\Gamma_A = \text{co } A$. In this setting we can choose $\beta_A((t_1, ..., t_n)) = \sum_{i=1}^n t_i x_i$, for every $(t_1, ..., t_n) \in \Sigma^{(n-1)}$. For every $z \in \text{co } A$, $(t_1(z), ..., t_n(z)) \in \beta_A^{-1}(z)$, $(\lambda_1, ..., \lambda_n) \in E^n$ with $\{(x_i, \lambda_i): i = 1, ..., n\} \subset \text{ipo } f$, by convexity of ipo f, we have $(z, \sum_{i=1}^n t_i(z) \lambda_i) = \sum_{i=1}^n t_i(z)(x_i, \lambda_i) \in \text{ipo } f$.

Now we prove some properties on H-concave or H^*-concave functions.

PROPOSITION 1. *Let $f: X \to E$ be a given function. If f is H-concave, then for every $\lambda \in E$ the set $\{x \in X: f(x) \in \lambda + \mathring{C}\}$ is H-convex in X. If f is H*-concave, then for every $\lambda \in E$ the set $\{x \in X: f(x) \nleq \lambda\}$ is H-convex in X.*

Proof. Let $\lambda \in E$. For an H-concave function f, we prove that the set $L_\lambda = \{x \in X: f(x) \in \lambda + \mathring{C}\}$ is H-convex. Let $A = \{x_1, ..., x_n\}$ be a finite subset of L_λ. Thus, $f(x_i) \in \lambda + \mathring{C}$ for every $i = 1, ..., n$, that is $\{(x_1, \lambda), ..., (x_n, \lambda)\} \subset \mathscr{I}_f$. By the H-concavity of f, the set $\bigcup_{z \in \Gamma_A} (z, \sum_{i=1}^n t_i(z) \lambda) = \bigcup_{z \in \Gamma_A} (z, \lambda) = \Gamma_A \times \{\lambda\}$ is contained in $\mathscr{I}_f$. Here $(t_1(z), ..., t_n(z)) \in \beta_A^{-1}(z)$ is an n-tuple compatible with the H-concavity of f. So for every $z \in \Gamma_A$, $f(z) \in \lambda + \mathring{C}$ or $\Gamma_A \subset L_\lambda$ as well.

The proof of the second statement is carried out by the same argument. The following proposition gives a characterization of the H-concavity.

PROPOSITION 2. *Let $f: X \to E$ be a given function. The following statements are equivalent:*

(i) *$f: X \to E$ is H-concave.*

(ii) *For every $A = \{x_1, ..., x_n\} \subset X$ and for every $z \in \Gamma_A$, there is an n-tuple $(t_1(z), ..., t_n(z)) \in \beta_A^{-1}(z)$ such that*

$$f(z) \geqslant \sum_{i=1}^n t_i(z) f(x_i). \tag{1}$$

Proof. (i) $\Rightarrow$ (ii). Let $A = \{x_1, ..., x_n\}$ be a finite subset of X and let $\gamma \in \mathring{C}$; so we have $\{(x_i, f(x_i) - \gamma),\ i = 1, ..., n\} \subset \mathscr{I}_f$. For every $z \in \Gamma_A$ and for every n-tuple $(t_1(z), ..., t_n(z)) \in \beta_A^{-1}(z)$ compatible with the H-concavity of f, we have

$$\left(z, \sum_{i=1}^{n} t_i(z)(f(x_i)-\gamma)\right) = \left(z, \sum_{i=1}^{n} t_i(z) f(x_i) - \gamma\right) \in \mathscr{I}_f.$$

Therefore, $f(z) \in (\sum_{i=1}^{n} t_i(z) f(x_i) - \gamma) + \mathring{C}$ and hence every $\gamma \in \mathring{C}$

$$f(z) > \sum_{i=1}^{n} t_i(z) f(x_i) - \gamma. \tag{2}$$

By the arbitrariness of γ, from (2) we obtain (1).

(ii) $\Rightarrow$ (i). By putting $A = \{x_1, ..., x_n\} \subset X$, let $\{(x_1, \lambda_1), ..., (x_n, \lambda_n)\} \subset A \times E$ be a finite subset of $\mathscr{I}_f$. Moreover, let $z \in \Gamma_A$. By (ii), there is an n-tuple $(t_1(z), ..., t_n(z)) \in \beta_A^{-1}(z)$ such that

$$f(z) \geqslant \sum_{i=1}^{n} t_i(z) f(x_i). \tag{3}$$

As $f(x_i) \in \lambda_i + \mathring{C}$ for every $i = 1, ..., n$, and by virtue of the convexity of $\mathring{C}$, $\sum_{i=1}^{n} t_i(z)(f(x_i) - \lambda_i) \in \mathring{C}$ or $\sum_{i=1}^{n} t_i(z) f(x_i) \in (\sum_{i=1}^{n} t_i(z)\lambda_i) + \mathring{C}$ as well. Therefore, by (3) $f(z) \in (\sum_{i=1}^{n} t_i(z)\lambda_i) + \mathring{C}$ and so (i) follows.

For the H^*-concave functions, we have the following

PROPOSITION 3. *If $f: X \to E$ is H^*-concave, then for every fixed $A = \{x_1, ..., x_n\} \subset X$ and for every $z \in \Gamma_A$ there is an n-tuple $(t_1(z), ..., t_n(z)) \in \beta_A^{-1}(z)$ such that*

$$f(z) \not< \sum_{i=1}^{n} t_i(z) f(x_i). \tag{1}$$

Proof. Let $A = \{x_1, ..., x_n\} \subset X$ and let $\gamma \in C \setminus \{0\}$ be fixed. The set $\{(x_i, f(x_i) - \gamma), i = 1, ..., n\}$ is contained in $\mathscr{I}_f^*$. For every $z \in \Gamma_A$ and for every n-tuple $(t_1(z), ..., t_n(z)) \in \beta_A^{-1}(z)$ compatible with the H^*-concavity of f, we have

$$\left(z, \sum_{i=1}^{n} t_i(z)(f(x_i)-\gamma)\right) = \left(z, \sum_{i=1}^{n} t_i(z) f(x_i) - \gamma\right) \in \mathscr{I}_f^*.$$

By the definition of $\mathscr{I}_f^*$, it follows that

$$f(z) \not\leqslant \sum_{i=1}^{n} t_i(z) f(x_i) - \gamma, \tag{2}$$

for every $\gamma \in C \setminus \{0\}$. For the arbitrariness of $\gamma \in \mathring{C} \setminus \{0\}$, we obtain (1).

We can remark that the proof of part (i)$\Rightarrow$(ii) in Proposition 2 as well as the proof of Proposition 3 points out that every n-tuple $(t_1(z), ..., t_n(z)) \in \beta_A^{-1}(z)$, $z \in \Gamma_A$, which is compatible with the H-concavity of f [respectively H^*-concavity of f] verifies (1) of Proposition 2 [respectively (1) of Proposition 3].

The next propositions concern some linear properties of H-concave or H^*-concave functions.

PROPOSITION 4. *Let $f, g: X \to E$ be two H-concave functions. We suppose that:*

for every finite subset $A \subset X$ and for every $z \in \Gamma_A$, there is an element in $\beta_A^{-1}(z)$ compatible with the H-concavity of f and of g. $(^+)$

Then, $f+g$ is H-concave.

Proof. The proof is an easy consequence of Proposition 2.

The following lemma allows us to prove a further linear property.

LEMMA 1. *Given $n+1$ points $\mu_0, ..., \mu_n \in E$ with $\mu_j \not\leqslant \mu_0$ for every $j = 1, ..., n$, there is $\omega \in E$ such that $\mu_j \not\leqslant \omega$, $j = 1, ..., n$, and $\mu_0 \in \omega - \mathring{C}$.*

Proof. We have to prove that the set $(\mu_0 + \mathring{C}) \cap (\bigcap_{j=1}^n (E \setminus (\mu_j + C)))$ is nonempty. Let us suppose that it is empty. Let $(\gamma_m)_{m \in \mathbb{N}}$ be a sequence in $\mu_0 + \mathring{C}$ converging to μ_0 (for example, we set $\gamma_m = \mu_0 + m^{-1}v$, where v is arbitrarily chosen in $\mathring{C}$). Since $\gamma_m \in \mu_0 + \mathring{C}$, we have $\gamma_m \in \bigcup_{j=1}^n (\mu_j + C)$ for every $m \in \mathbb{N}$. By the closedness of $\mu_j + C$, it follows that $\mu_0 \in \bigcup_{i=1}^n (\mu_j + C)$. Hence there is an integer k, $1 \leqslant k \leqslant n$ such that $\mu_0 \geqslant \mu_k$, which contradicts the assumptions.

PROPOSITION 5. *Let $f, g: X \to E$ be given functions. Let us suppose that f is H^*-concave, g is H-concave, and moreover the following property holds:*

For every finite subset $A \subset X$ and for every $z \in \Gamma_A$ there is an element in $\beta_A^{-1}(z)$ compatible with the H^*-concavity of f and with the H-concavity of g. $(^{++})$

Then, $f+g$ is H^*-concave.

Proof. Let $A = \{x_1, ..., x_n\}$ be a finite subset of X and let $(\lambda_1, ..., \lambda_n) \in E^n$ such that the set $\{(x_1, \lambda_1), ..., (x_n, \lambda_n)\} \subset \mathscr{I}^*_{f+g}$. Therefore, $f(x_i) \not\leqslant \lambda_i - g(x_i)$ for every $i = 1, ..., n$. Now, we fix $i \in \{1, ..., n\}$. By virtue of Lemma 1 for $\mu_0 = \lambda_i - g(x_i)$ and $\mu_1 = f(x_i)$, there is ω_i such that $f(x_i) \not\leqslant \omega_i$ and $\lambda_i - g(x_i) \in \omega_i - \mathring{C}$. So, $f(x_i) \not\leqslant \omega_i$ and $g(x_i) \in (\lambda_i - \omega_i) + \mathring{C}$. By the definitions of $\mathscr{I}^*_f$ and $\mathscr{I}_g$ it follows that $\{(x_i, \omega_i), i = 1, ..., n\} \subset \mathscr{I}^*_f$ and

$\{(x_i, \lambda_i - \omega_i),\ i = 1, ..., n\} \subset \mathcal{J}_g$. By assumed hypotheses, for every $z \in \Gamma_A$ there is an n-tuple $(t_1(z), ..., t_n(z)) \in \beta_A^{-1}(z)$ such that

$$\left(z, \sum_{i=1}^{n} t_i(z)\omega_i\right) \in \mathcal{J}_f^*$$

$$\left(z, \sum_{i=1}^{n} t_i(z)(\lambda_i - \omega_i)\right) \in \mathcal{J}_g.$$

Thus for every $z \in \Gamma_A$ we have $f(z) \nleq \sum_{i=1}^{n} t_i(z)\omega_i$ and $g(z) \in (\sum_{i=1}^{n} t_i(z)(\lambda_i - \omega_i) + \mathring{C})$, that is

$$f(z) + g(z) \nleq \sum_{i=1}^{n} t_i(z)\omega_i + \sum_{i=1}^{n} t_i(z)(\lambda_i - \omega_i) = \sum_{i=1}^{n} t_i(z)\lambda_i.$$

The proof is now complete.

3. Lower Semicontinuity: Characterization and Properties

In this section we adjoin a greatest element $+\infty$ to E and we extend the linear operations on E in a natural way. So, a smallest element $-\infty$ is adjoined to E, too. In the enlarged space $E \cup \{+\infty\}$, the positive cone C will be thought as containing $+\infty$. The family of sets $\{\lambda + \mathring{C} : \lambda \in E\}$ will be a natural neighborhood-base at $+\infty$.

Moreover, we assume that E is an order-complete Riesz space [14].

Proposition 6. *Let $f: X \to E$ be a given function. The following statements are equivalent*:

(i) *For every $\lambda \in E$ the set $\{x \in X : f(x) \in \lambda + \mathring{C}\}$ is open*;

(ii) *For every $\mu \in \mathring{C}$ and for every $x_0 \in X$ there is a neighborhood $\mathcal{U}_{x_0}$ in X such that*

$$f(y) + \mu - f(x_0) \in \mathring{C}$$

for every $y \in \mathcal{U}_{x_0}$.

Proof. The proof is carried out as in Theorem 5.1 of [23].

Definition 2. A function $f: X \to E$ verifying the property (i) or (ii) in Proposition 6 will be called *lower semicontinuous* (see [23]).

As an easy consequence of Proposition 6, the following linear property of lower semicontinuous functions holds.

PROPOSITION 7. *If $f, g: X \to E$ are lower semicontinuous functions, then $f+g$ is lower semicontinuous, too.*

Next, we state a further property of lower semicontinuity, which will be used in what follows.

We premise a lemma.

LEMMA 2. *Let $f: X \to E$ be a lower semicontinuous function. If $(x_\alpha)_{\alpha \in I}$ is a net converging to x_0 in X, then we have*

$$\sup_{\bar{\alpha} \in I} \inf_{\alpha \geqslant \bar{\alpha}} f(x_\alpha) \geqslant f(x_0).$$

Proof. Let us remark that the existence of the element $+\infty$ and the order completeness of E justify the sup and inf operations. The proof is the same as in Lemma 5.1 in [23].

PROPOSITION 8. *Let $f: X \to E$ be a lower semicontinuous function. Then for every $\lambda \in E$, the set $\{x \in X: f(x) \leqslant \lambda\}$ is closed in X.*

Proof. Let $\lambda \in E$ and let $(x_\alpha)_{\alpha \in I}$ be a net in X converging to x_0 such that $f(x_\alpha) \leqslant \lambda$ for every $\alpha \in I$. Therefore $\sup_{\bar{\alpha} \in I} \inf_{\alpha \geqslant \bar{\alpha}} f(x_\alpha) \leqslant \lambda$ and so the thesis follows by Lemma 2.

4. INEQUALITIES IN RIESZ SPACES

In this section we state some theorems on the variational type inequalities. Because of our abstract setting (H-convex structure, relaxed compactness hypothesis, functions taking value in the Riesz spaces), our results generalize some recent theorems by Lassonde [22] and a fortiori some classical results obtained by Debrunner and Flor [5] and Hartman and Stampacchia [16].

In order to apply the linear properties of the H-concave or H^*-concave functions (see Propositions 4, 5 of Section 2), from now on we suitably assume that the compatibility assumptions $(^+)$ and $(^{++})$ are satisfied, without mentioning them explicitly.

The next two theorems are proved as an application of the following generalization of the KKM theorem stated in [2].

THEOREM 1. *Let $(X, \{\Gamma_A\})$ be an H-space and $F: X \to X$ an H-KKM multifunction such that:*

(a) *For each $x \in X$, $F(x)$ is compactly closed.*

(b) *There is a compact set $L \subset X$ and an H-compact $K \subset X$ such that, for each H-convex set D with $K \subset D \subset X$, we have $\bigcap_{x \in D}(F(x) \cap D) \subset L$. Then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

THEOREM 2. *Let $(X, \{\Gamma_A\})$ be an H-space and let $\phi: X \to E$, $f: X \times X \to E$ be two function with the properties:*

(a) *$-\phi$ is H-concave in X.*

(b) *For every $\lambda \in E$, the set $\{x \in X: \phi(x) \in \lambda + \mathring{C}\}$ is compactly open.*

(c) *For every $\lambda \in E$ and for every $x \in X$, the set $\{y \in X: f(x, y) \in \lambda + \mathring{C}\}$ is compactly open.*

(d) *For every $y \in X$, the function $x \to f(x, y)$ is H^*-concave.*

(e) *There are a compact set L and an H-compact $K \subset X$ such that for every $y \in X \backslash L$ there is $x \in K_y$ with $f(x,y) + \phi(y) \nleq \phi(x)$.*

Then the following alternative holds:

(1) *there is $y_0 \in X$ such that $f(x, y_0) + \phi(y_0) \leqslant \phi(x)$ for every $x \in X$ or*

(2) *there is $x_0 \in X$ such that $f(x_0, x_0) \nleq 0$.*

Proof. Let us define the multifunction $F: X \to X$ by $F(x) = \{y \in X: f(x, y) + \phi(y) \leqslant \phi(x)\}$. If there is $x_0 \in X$ such that $x_0 \notin F(x_0)$, then $f(x_0, x_0) \nleq 0$ and so we have (2). Otherwise, $x \in F(x)$ for every $x \in X$. In order to obtain the thesis, we prove that F satisfies the assumptions of Theorem 1. If F is not H-KKM, there is a finite subset $A \subset X$ such that $\Gamma_A \not\subset \bigcup_{x \in A} F(x)$ and so there exists $y \in \Gamma_A$ with $y \notin F(x)$ for every $x \in A$. This means that $A \subset F^*(y) = \{x \in X: \ f(x, y) + \phi(y) \nleq \phi(x)\}$. By Assumptions (a), (d) and Propositions 5, 1, the set $F^*(y)$ is H-convex. Therefore, $\Gamma_A \subset F^*(y)$ and hence $f(y, y) + \phi(y) \nleq \phi(y)$, that is $f(y, y) \nleq 0$, and that is a contradiction. Thus, F is an H-KKM multifunction. Moreover, by virtue of hypotheses (b), (c) and Propositions 7, 8, it is easy to show that $F(x)$ is compactly closed for every $x \in X$. Relative to hypothesis (b) of Theorem 1, let L and K be the subsets of X specified by (e) and let D be an H-convex subset of X containing K. If $\bigcap_{x \in D}\{F(x) \cap D\} \not\subset L$ holds, then there is $y \in D$ such that $y \notin L$ and $f(x, y) + \phi(y) \leqslant \phi(x)$ for every $x \in D$, which is a contradiction. Finally, by Theorem 1 we have $\bigcap_{x \in X} F(x) \neq \emptyset$, which is part (1) of the alternative.

In the following Theorems 3, 5, 6, the set of conditions (a), (b), (c) in their various versions represents (also in the case $E = \mathbb{R}$) an extension of the Lassonde semimonotonicity [22].

THEOREM 3. *Let $(X, \{\Gamma_A\})$ be an H-space and let $f: X \times X \times X \to E$ be a function verifying the following conditions:*

(a) *For every $(\xi, x, y) \in X \times X \times X$, $f(\xi, x, y) + f(\xi, y, x) \geqslant 0$.*

(b) *For every* $(\xi, x) \in X \times X$, *the set* $\{y \in X : f(\xi, x, y) \nleq 0\}$ *is H-convex.*

(c) *For every* $x \in X$, *the set* $\{y : f(y, x, y) \geqslant 0\}$ *is compactly closed.*

(d) *For every* $x \in X$, $f(x, x, x) = 0$.

(e) *There are a compact subset* $L \subset X$ *and an H-compact* $K \subset X$ *such that for every* $y \in X \backslash L$ *there is* $x \in K_y$ *with* $f(y, x, y) \ngeqslant 0$.

Then there is $y_0 \in X$ *such that*

$$f(y_0, x, y_0) \geqslant 0$$

for every $x \in X$.

Proof. For every $x \in X$ we define $F(x) = \{y \in X : f(y, x, y) \geqslant 0\}$. The theorem is proved by using Theorem 1 for the multifunction F. At first, we show that F is H-KKM. If that is not the case, there is a finite subset $A \subset X$ such that $\Gamma_A \not\subset \bigcup_{x \in A} F(x)$ and hence there exists $y \in \Gamma_A$ with $y \notin F(x)$, for every $x \in A$. Therefore $A \subset F^*(y) = \{x \in X : f(y, x, y) \ngeqslant 0\} \subset \{x \in X : f(y, y, x) \nleq 0\}$. Here, the last inclusion is a consequence of the hypothesis (a). By (b) it follows that $\Gamma_A \subset \{x \in X : f(y, y, x) \nleq 0\}$ and so $f(y, y, y) \nleq 0$, which contradicts the hypothesis (d). The hypothesis (c) affirms that the set $F(x)$ is compactly closed for every $x \in X$. Finally we prove that the hypothesis (b) of Theorem 1 is verified. Let K, L be the sets specified by (e). Let us suppose that there is an H-convex set $D \subset X$ such that $K \subset D$ and $\bigcap_{x \in D}(F(x) \cap D) \not\subset L$. Therefore there is $y \in D$, $y \in X \backslash L$, and $f(y, x, y) \geqslant 0$ for every $x \in D$. As $K \cup \{y\} \subset D$ this contradicts (e). By virtue of Theorem 1 we have $\bigcap_{x \in X} F(x) \neq \varnothing$, i.e., the thesis.

Theorem 4, which is a slight extension of Theorem 1, proves some further results on the perturbed variational inequalities. Moreover it represents a generalization of Theorem II in [22].

THEOREM 4. *Let* $(X, \Gamma_A\})$ *be an H-space, and let* $F, G : X \to X$ *be two multifunctions such that*:

(a) *For every* $x \in X$, $G(x)$ *is compactly closed and* $F(x) \subset G(x)$.

(b) F *is an H-KKM multifunction.*

(c) *There are a compact* $L \subset X$ *and an H-compact* $K \subset X$ *such that, for every H-convex set D with* $K \subset D \subset X$, *we have* $\bigcap_{x \in D}(F(x) \cap D) \subset L$.

(d) *For every H-convex D with* $K \subset D \subset X$, *we have* $\bigcap_{x \in D}(G(x) \cap D) \neq \varnothing$ *if and only if* $\bigcap_{x \in D}(F(x) \cap D) \neq \varnothing$.

Then $\bigcap_{x \in X} F(x) \neq \varnothing$.

Proof. By virtue of (d) with $D = X$, it is sufficient to prove that $\bigcap_{x \in X}(G(x) \cap L) \neq \varnothing$. As in Theorem 1 it is enough to show that $\bigcap_{x \in A}(G(x) \cap L) \neq \varnothing$ for every finite subset $A \subset X$. Let A be a finite subset and let $X_0 \subset X$ be both a compact and H-convex set such that $K \cup A \subset X_0$.

By (c) we have $\bigcap_{x\in X_0}(F(x)\cap X_0)\subset L$ and thus $\bigcap_{x\in X_0}(F(x)\cap X_0)\subset \bigcap_{x\in A}(F(x)\cap L)\subset\bigcap_{x\in A}(G(x)\cap L)$. Now, bearing assumption (d) in mind, it is sufficient to prove that $\bigcap_{x\in X_0}(G(x)\cap X_0)\neq\varnothing$. Let us consider the multifunction $G_0\colon X_0\to X_0$ defined by $G_0(x)=G(x)\cap X_0$. The H-KKM property of G (see assumptions (a) and (b)) implies the same property on G_0 with respect to the H-space $(X_0, \{\Gamma_{A\cap X_0}\})$. Through the closedness of $G_0(x)$ in the compact X_0 and by using Corollary 1 in [19], we deduce that $\bigcap_{x\in X_0}G_0(x)=\bigcap_{x\in X_0}(G(x)\cap X_0)\neq\varnothing$ and so the proof is complete.

The next theorems, whose proofs are based on Theorem 4, extend Theorem 2.4 by Lassonde [22] in various directions.

In Theorem 5 the following property on the function $f\colon X\times X\times X\to E$ will be assumed:

> If for every $(\xi, x)\in X\times X$ there are $\lambda_1\in E$ and $\lambda_2\in C$ such that $f(\xi, z, x)-t_1(z)\lambda_1-t_2(z)\lambda_2\notin\mathring{C}$ for every $z\in\Gamma_{\{\xi,x\}}\setminus\{\xi\}$ and for some $(t_1(z), t_2(z))\in\beta^{-1}_{\{\xi,x\}}(z)$, then $f(\xi,\xi,x)-\lambda_1-\lambda_2\notin\mathring{C}$. (°)

Also in the Lassonde setting [22] (X convex space, $\Gamma_A=\operatorname{co}A$, $E=\mathbb{R}$) this property generalizes the definition of hemicontinuity due to Lassonde [22].

Indeed, for every fixed $(\xi, x)\in X\times X$ let $h\colon[0,1]\to\mathbb{R}$ be the function defined by $h(t)=f(\xi,(1-t)\xi+tx, x)$. If h is lower semicontinuous at $t=0$, we have $h(0)=f(\xi,\xi,x)\leqslant\underline{\lim}_{t\to0}h(t)$. Let us suppose that there exist $\lambda_1,\lambda_2\in\mathbb{R}$, $\lambda_2\geqslant0$ with $h(t)\leqslant(1-t)\lambda_1+t\lambda_2$ for every $t\in[0,1]$. Then $f(\xi,\xi,x)\leqslant\underline{\lim}_{t\to0}h(t)\leqslant\underline{\lim}_{t\to0}(1-t)\lambda_1+t\lambda_2\leqslant\lambda_1+\lambda_2$.

THEOREM 5. *Let $(X,\Gamma_A\})$ be an H-space and let $\phi\colon X\to E$, $f\colon X\times X\times X\to E$ be two functions such that:*

(a) *For every $(\xi,x,y)\in X\times X\times X$, $f(\xi,x,y)+f(\xi,y,x)\geqslant0$.*

(b) *For every $(\xi,x)\in X\times X$, the function $y\to f(\xi,x,y)$ is H-concave.*

(c) *For every $x\in X$ and for every $\lambda\in E$, the set $\{y\in X\colon f(y,x,y)\in\lambda-\mathring{C}\}$ is compactly open.*

(d) *For every $(\xi,x)\in X\times X$, $f(\xi,x,x)\leqslant0$.*

(e) *The function f satisfies the property (°).*

(f) *The function $-\phi$ is H-concave.*

(g) *For every $\lambda\in E$, the set $\{x\in X\colon\phi(x)\in\lambda+\mathring{C}\}$ is compactly open.*

(h) *There are a compact $L\subset X$ and an H-compact $K\subset X$ such that for every $y\in X\setminus L$ there is $x\in K_y$ with $f(y,y,x)+\phi(y)-\phi(x)\in\mathring{C}$.*

Then, there exists $y_0\in X$ such that

$$f(y_0,y_0,x)+\phi(y_0)-\phi(x)\notin\mathring{C}$$

for every $x\in X$.

Proof. For every $x \in X$ we define $F(x) = \{y \in X : f(y, y, x) + \phi(y) - \phi(x) \notin \mathring{C}\}$. The multifunction F is H-KKM. Indeed, if there is a finite subset $A \subset X$ such that $\Gamma_A \not\subset \bigcup_{x \in X} F(x)$, then there exists $y \in \Gamma_A$ with $y \notin F(x)$ for every $x \in A$. By virtue of hypotheses (b) and (f) and making use of Propositions 4 and 1 we have $y \in F^*(y) = \{x \in X : f(y, y, x) + \phi(y) - \phi(x) \in \mathring{C}\}$, which contradicts (d). Let us define $G(x) = \{y \in X : f(y, x, y) + \phi(x) - \phi(y) \notin -\mathring{C}\}$. By (a) we have $F(x) \subset G(x)$ for every $x \in X$. Moreover, the assumptions (c), (g) and Proposition 7 imply that $G(x)$ is compactly closed for every $x \in X$.

Now, we prove that $F(x)$ satisfies the property (c) of Theorem 4. If there exists an H-convex D with $K \subset D \subset X$ such that $\bigcap_{x \in D} (F(x) \cap D) \not\subset L$, then there is $y \in X \backslash L$ and $y \in F(x)$ for every $x \in D$. This contradicts (h).

Finally, we prove the condition (d) of Theorem 4. Let D be an H-convex set in X. As $F(x) \subset G(x)$ for every $x \in X$, it is sufficient to prove that $\bigcap_{x \in D} (G(x) \cap D) \subset \bigcap_{x \in D} (F(x) \cap D)$. Let $y_0 \in \bigcap_{x \in D} (G(x) \cap D)$ and let $x \in D$. As D is H-convex we have $\Gamma_{\{x, y_0\}} \subset D$ and hence $y_0 \in G(z)$ for every $z \in \Gamma_{\{x, y_0\}}$ or

$$f(y_0, z, y_0) + \phi(z) - \phi(y_0) \notin -\mathring{C}. \tag{1}$$

By (f), bearing Proposition 2 in mind, for every $z \in \Gamma_{\{x, y_0\}}$ there is $t(z) \in [0, 1]$ such that

$$\phi(z) \leqslant t(z)\, \phi(x) + (1 - t(z))\, \phi(y_0). \tag{2}$$

By (1) for every $z \in \Gamma_{\{x, y_0\}}$ we have

$$f(y_0, z, y_0) - t(z)\, \phi(y_0) + t(z)\, \phi(x) \notin -\mathring{C}. \tag{3}$$

Besides (3), by (d) and (b) for every $z \in \Gamma_{\{x, y_0\}}$ we obtain

$$0 \geqslant f(y_0, z, z) \geqslant t(z) f(y_0, z, x) + (1 - t(z)) f(y_0, z, y_0) \tag{4}$$

and hence by applying (3), we have

$$f(y_0, z, x) + (1 - t(z))[\phi(y_0) - \phi(x)] \notin \mathring{C}. \tag{5}$$

Indeed, by (3) it follows that $(1 - t(z))[f(y_0, z, y_0) - t(z)(\phi(y_0) - \phi(x))] \notin -\mathring{C}$ and so

$$(1 - t(z)) f(y_0, z, y_0) \notin (1 - t(z))\, t(z)(\phi(y_0) - \phi(x)) - \mathring{C}.$$

By (4), $(1 - t(z)) f(y_0, z, y_0) \leqslant -t(z) f(y_0, z, x)$ and hence for every $z \in \Gamma_{\{x, y_0\}}$,

$$-t(z)[f(y_0, z, x) + (1 - t(z))(\phi(y_0) - \phi(x))] \notin -\mathring{C},$$

that is

$$f(y_0, z, x) + (1 - t(z))(\phi(y_0) - \phi(x)) \notin \mathring{C}$$

for every $z \in \Gamma_{\{x, y_0\}} \setminus \{y_0\}$. Note that if $z \neq y_0$, then $t(z) \neq 0$. By putting $\lambda_1 = \phi(x) - \phi(y_0)$ and $\lambda_2 = 0$, by (e) we have

$$f(y_0, y_0, x) + \phi(y_0) - \phi(x) \notin \mathring{C}$$

for every $x \in D$ or $y_0 \in \bigcap_{x \in D} (F(x) \cap D)$. The proof is now complete by using Theorem 4.

As the range of functions is an abstract topological Riesz space, by suitably modifying the assumptions, we may state a different version of Theorem 5. However, the new formulation is the same as Theorem 5 in the case $E = \mathbb{R}$.

The property (°) is now replaced by

> If for every $(\xi, x) \in X \times X$ there are $\lambda_1 \in E$ and $\lambda_2 \in C$ such that $f(\xi, z, x) - t_1(z)\lambda_1 - t_2(z)\lambda_2 \leqslant 0$ for every $z \in \Gamma_{\{\xi, x\}} \setminus \{\xi\}$ and for some $(t_1(z), t_2(z)) \in \beta^{-1}_{\{\xi, x\}}(z)$ then we have $f(\xi, \xi, x) - \lambda_1 - \lambda_2 \leqslant 0$. (°°)

THEOREM 6. *Let $(X, \{\Gamma_A\})$ be an H-space and let $\phi: X \to E$, $f: X \times X \times X \to E$ be two functions such that*:

(a) *For every $(\xi, x, y) \in X \times X \times X$, $f(\xi, x, y) + f(\xi, y, x) \geqslant 0$.*

(b) *For every $(\xi, x) \in X \times X$, the function $y \to f(\xi, x, y)$ is H-concave.*

(c) *For every $x \in X$ and for every $\lambda \in E$, the set $\{y \in X: f(y, x, y) \in \lambda - \mathring{C}\}$ is compactly open.*

(d) *For every $(\xi, x) \in X \times X$, $f(\xi, x, x) \leqslant 0$.*

(e) *The function f satisfies property* (°°).

(f) *The function $-\phi$ is H-concave.*

(g) *For every $\lambda \in E$, the set $\{x \in X: \phi(x) \in \lambda + \mathring{C}\}$ is compactly open.*

(h) *There are a compact $L \subset X$ and an H-compact $K \subset X$ such that for every $y \in X \setminus L$ there is $x \in K_y$ such that $f(y, y, x) + \phi(y) \nleqslant \phi(x)$.*

(i) *For every $y \in X$, the set $\{x \in X: f(y, y, x) + \phi(y) \nleqslant \phi(x)\}$ is H-convex.*

Then, there is $y_0 \in X$ such that

$$f(y_0, y_0, x) + \phi(y_0) \leqslant \phi(x)$$

for every $x \in X$.

Proof. For every $x \in X$ we set $F(x) = \{y \in X: f(y, y, x) + \phi(y) \leqslant \phi(x)\}$ and $G(x) = \{y \in X: f(y, x, y) + \phi(x) \geqslant \phi(y)\}$. Now, the proof is carried out as in Theorem 5. We only remark that the assumption (i) is employed to prove the H-KKM property on multifunction F.

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